

# A MULTI-VARIATE GENERATING FUNCTION FOR THE WEYL DIMENSION FORMULA

WAYNE JOHNSON

**ABSTRACT.** We present a closed form for a multi-variate generating function for the dimensions of the irreducible representations of a semisimple, simply connected linear algebraic group over  $\mathbb{C}$  whose highest weights lie in a finitely generated lattice cone in the dominant chamber. This result generalizes the formula for the Hilbert series of an equivariant embedding of a homogeneous projective variety. As a special case, we show how the multi-variate series can be used to compute the Hilbert series of the determinantal varieties.

## 1. INTRODUCTION

Let  $G$  be a semisimple, simply connected linear algebraic group over  $\mathbb{C}$ , and fix a choice  $T \subset B \subset G$  of maximal torus and Borel subgroup. The choice of Borel gives us a set of positive roots  $\Phi^+$  for  $\mathfrak{g} := \text{Lie}(G)$ , and a set  $P_+(\mathfrak{g})$  of dominant integral weights for  $\mathfrak{g}$ . To each  $\lambda \in P_+(\mathfrak{g})$ , the Theorem of the Highest Weight gives us a finite dimensional irreducible representation  $L(\lambda)$  of  $G$ . Using this representation, we can find a parabolic subgroup  $P \supset B$ —namely,  $P$  is the subgroup of  $G$  that stabilizes the unique hyperplane  $H$  in  $L(\lambda)$  fixed by  $B$ . Then we get an embedding of  $G/P$  into the projective space of all hyperplanes in  $L(\lambda)$ , denoted  $\mathbb{P}(L(\lambda))$ , given by  $\pi_\lambda(gP) := g(H)$ .

In the paper “On the Hilbert polynomials and Hilbert series of homogeneous projective varieties” [6], the authors present a closed form for the Hilbert series of the equivariant embedding  $\pi_\lambda$  of  $G/P$  into  $\mathbb{P}(L(\lambda))$ . In particular, they show that the homogeneous coordinate ring  $A(G/P)$  of such an embedding is isomorphic to  $\bigoplus_{n \in \mathbb{N}} L(n\lambda)$ , where  $L(\lambda)$  denotes the above irreducible representation of  $G$  with highest weight  $\lambda$ . The Hilbert series of the embedding is then given by the function

$$HS_q(\lambda) = \sum_{n \in \mathbb{N}} \dim(L(n\lambda))q^n.$$

They then prove the Hilbert series has the following closed form.

**Theorem (Gross and Wallach).** *The Hilbert series of the embedding  $\pi_\lambda$  of  $G/P$  is*

$$\prod_{\alpha \in \Phi^+} \left( \frac{(\lambda, \alpha)}{(\rho, \alpha)} q \frac{d}{dq} + 1 \right) \frac{1}{1 - q}.$$

A natural generalization of the above Hilbert series is the formal power series

$$(1.1) \quad HS_{\mathbf{q}}\langle \lambda_1, \dots, \lambda_k \rangle := \sum_{(a_1, \dots, a_k) \in \mathbb{N}^k} \dim(L(a_1\lambda_1 + \dots + a_k\lambda_k))q_1^{a_1} \dots q_k^{a_k},$$

where  $q_1, \dots, q_k$  are indeterminates, and  $\lambda_1, \dots, \lambda_k$  are dominant integral weights. The main result of this paper is to prove a generalization of the above theorem and find a closed form of (1.1). We prove the following.

**Main Theorem.** *Let  $\lambda_1, \dots, \lambda_k$  be dominant integral weights. Then*

$$(1.2) \quad HS_q\langle\lambda_1, \dots, \lambda_k\rangle = \prod_{\alpha \in \Phi^+} \left( 1 + c_{\lambda_1}(\alpha)q_1 \frac{\partial}{\partial q_1} + \dots + c_{\lambda_k}(\alpha)q_k \frac{\partial}{\partial q_k} \right) \prod_{i=1}^k \frac{1}{1 - q_i},$$

where  $c_\lambda(\alpha) := \frac{(\lambda, \alpha)}{(\rho, \alpha)}$ .

This is a generating function for the dimensions of the finite dimensional irreducible representations of  $G$  whose highest weight lies in the lattice cone in  $P_+(\mathfrak{g})$  generated by  $\lambda_1, \dots, \lambda_k$ . We denote such a lattice cone by  $\langle\lambda_1, \dots, \lambda_k\rangle$ . Note that in [2], the authors give the special case where  $\mathfrak{g}$  has rank  $k$  and we choose  $\lambda_i$  to be the fundamental dominant weight  $\omega_i$  for  $1 \leq i \leq k$ . The above theorem applies to a more general lattice cone.

Note that (1.2) allows us to compute the Hilbert series for many varieties by first computing the multi-variate series and then specializing to a gradation on the algebra

$$\bigoplus_{\lambda \in \langle\lambda_1, \dots, \lambda_k\rangle} L(\lambda),$$

via a suitable substitution. This is especially useful in computing the Hilbert series of determinantal varieties, which are traditionally quite difficult to compute (see, for example, [3]). The series (1.2) is not difficult to compute using Mathematica or Maple, and then we find the Hilbert series by specializing the grade appropriately. For instance, in §4, we give a linear recursion on  $n$  for computing (1.2) for the weights in  $\langle 2\omega_1, 2\omega_2 \rangle$ , where  $\omega_1$  and  $\omega_2$  are the first two fundamental dominant weights of  $SL(n, \mathbb{C})$ , and then specialize this two variable series to obtain the Hilbert series of the determinantal variety of rank at most two symmetric matrices in  $M_n(\mathbb{C})$ . The methods presented in this paper bypass much of the complicated machinery traditionally used to compute these series.

## 2. PRELIMINARIES

Throughout this paper, let  $G$  be a semisimple, simply connected linear algebraic group over  $\mathbb{C}$ . Let  $T$  be a maximal torus and  $T \subset B \subset G$  a choice of Borel subgroup containing  $T$ . Let  $U$  be the unipotent radical of  $B$ . We denote by  $\mathfrak{g}, \mathfrak{h}$ , and  $\mathfrak{b}$  the Lie algebras of  $G, T$ , and  $B$ , respectively. Let  $\Phi$  be the root system given by the pair  $(\mathfrak{g}, \mathfrak{h})$ , and let  $\Phi^+$  denote the set of positive roots corresponding to  $\mathfrak{b}$ . Throughout this paper, we set  $d := |\Phi^+|$ .

Let  $P_+(\mathfrak{g})$  denote the set of dominant integral weights. To each weight  $\lambda \in P_+(\mathfrak{g})$ , let  $L(\lambda)$  denote the irreducible representation of  $G$  with highest weight  $\lambda$ , and denote by  $(,)$  the non-degenerate bilinear form on  $\mathfrak{h}^*$  induced by the Killing form. Then the following is well known (see, for example, p.336 in [5]).

**Weyl Dimension Formula.** *Let  $\lambda \in P_+(\mathfrak{g})$ . Then*

$$\dim(L(\lambda)) = \prod_{\alpha \in \Phi^+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)},$$

where  $\rho$  denotes  $\frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ .

Following the notation in [6], let  $c_\lambda(\alpha) := \frac{(\lambda, \alpha)}{(\rho, \alpha)}$ . Then the above formula can be written as  $\dim(L(\lambda)) = \prod_{\alpha \in \Phi^+} (c_\lambda(\alpha) + 1)$ .

Given a graded  $\mathbb{C}$ -algebra  $A$  with  $i$ th homogeneous component  $A_i$ , we define its *Hilbert function* to be the map  $HF_A : \mathbb{N} \rightarrow \mathbb{N}$ , given by  $HF_A(i) = \dim(A_i)$ . Then the *Hilbert series* of  $A$  is the formal power series

$$HS_q(A) := \sum_{n \in \mathbb{N}} HF_A(n) q^n.$$

We give some basic properties of the Hilbert function and series for a graded  $\mathbb{C}$ -algebra  $A$ . For further information, see, for example, [1],[7]. If  $A$  is generated by  $A_1$ , then the Hilbert series of  $A$  must represent a rational function of the form

$$\frac{p(q)}{(1-q)^d},$$

where  $p(q) \in \mathbb{Z}[q]$  is a polynomial in  $q$  with integer coefficients. Further, if we consider the variety given by the spectrum of  $A$ , then the dimension of this variety is  $d$ .

As a generalization of the above, if we have an  $\mathbb{N}^k$ -graded  $\mathbb{C}$ -algebra  $A$  with homogeneous component  $A_{(a_1, \dots, a_k)}$  corresponding to the element  $(a_1, \dots, a_k) \in \mathbb{N}^k$ , we can define its  $\mathbb{N}^k$ -graded *Hilbert series* as the formal powers series

$$\sum_{(a_1, \dots, a_k) \in \mathbb{N}^k} \dim(A_{(a_1, \dots, a_k)}) q_1^{a_1} \dots q_k^{a_k}.$$

This series can be restricted via a substitution to a single grading on  $A$ . For example, we could make the substitution  $q_i \mapsto q$  to get a Hilbert series for  $A$ . Note that different restrictions correspond to different gradations of  $A$ , and these may give different Hilbert series.

In [6], the authors are interested in computing the Hilbert series of the homogeneous coordinate ring of an equivariant embedding of  $G/P$  into a projective space. We recall the details. Given any irreducible highest weight representation  $L(\lambda)$  of  $G$ , we can consider the parabolic subgroup given by the stabilizer of the unique hyperplane  $H$  in  $L(\lambda)$  fixed by the Borel subgroup  $B$ . If we denote by  $\mathbb{P}(L(\lambda))$  the projective space of all hyperplanes in  $L(\lambda)$ , then we have an embedding

$$\pi_\lambda : G/P \rightarrow \mathbb{P}(L(\lambda)),$$

given by the formula  $\pi_\lambda(gP) := g(H)$ . Then it is a consequence of the Borel-Weil theorem that the homogeneous coordinate ring  $A_\lambda(G/P)$  is a sum of highest weight representations. Namely,

$$A_\lambda(G/P) = \bigoplus_{n \in \mathbb{N}} L(n\lambda).$$

Thus, the Hilbert series of the embedding is  $\sum_{n \in \mathbb{N}} \dim(L(n\lambda)) q^n$ . The authors then prove that this series has the following closed form.

**Theorem (Gross and Wallach).** *The Hilbert series of the embedding  $\pi_\lambda$  of  $G/P$  is*

$$\prod_{\alpha \in \Phi^+} \left( c_\lambda(\alpha) q \frac{d}{dq} + 1 \right) \frac{1}{1-q}.$$

We want to extend this to a multi-variate series graded over finitely many dominant integral weights. To this end, we use the notation  $\mathbf{a}$  for a  $k$ -tuple  $(a_1, \dots, a_k) \in \mathbb{N}^k$ . We use the convention that  $\mathbf{a}^{\mathbf{i}} := a_1^{i_1} \dots a_k^{i_k}$  for two  $k$ -tuples  $\mathbf{a}$  and  $\mathbf{i}$ . We denote by  $|\mathbf{i}|$  the sum of the indices  $i_1 + \dots + i_k$ . Given partial derivatives  $\frac{\partial}{\partial q_i}$  and a  $k$ -tuple  $\mathbf{i}$ , we use the notation  $\left( \frac{\partial}{\partial \mathbf{q}} \right)^{\mathbf{i}}$  for the product  $\left( \frac{\partial}{\partial q_1} \right)^{i_1} \dots \left( \frac{\partial}{\partial q_k} \right)^{i_k}$ .

We use the notation  $\langle \lambda_1, \dots, \lambda_k \rangle$  to denote the lattice cone in the dominant chamber generated by the dominant integral weights  $\lambda_1, \dots, \lambda_k$ , and consider the following series.

$$HS_{\mathbf{q}} \langle \lambda_1, \dots, \lambda_k \rangle := \sum_{\mathbf{a} \in \mathbb{N}^k} \dim(L(a_1 \lambda_1 + \dots + a_k \lambda_k)) \mathbf{q}^{\mathbf{a}}$$

We prove the following.

**Main Theorem.** *Let  $\lambda_1, \dots, \lambda_k$  be dominant integral weights. Then*

$$(2.1) \quad HS_{\mathbf{q}} \langle \lambda_1, \dots, \lambda_k \rangle = \prod_{\alpha \in \Phi^+} \left( 1 + c_{\lambda_1}(\alpha) q_1 \frac{\partial}{\partial q_1} + \dots + c_{\lambda_k}(\alpha) q_k \frac{\partial}{\partial q_k} \right) \prod_{i=1}^k \frac{1}{1-q_i},$$

$$\text{where } c_\lambda(\alpha) := \frac{(\lambda, \alpha)}{(\rho, \alpha)}$$

We will use the above formula to compute the Hilbert series for certain determinantal varieties. To this end, we define the *determinantal variety of rank  $k$*  to be the subset of all rank at most  $k$  matrices in  $M_{m,n}(\mathbb{C})$ . The *symmetric determinantal variety of rank  $k$*  is the subset of rank at most  $k$  matrices in  $Sym_n(\mathbb{C}) := \{X \in M_n(\mathbb{C}) \mid X - X^T = 0\}$ . The *anti-symmetric determinantal variety of rank  $2k$*  is the subset of rank at most  $2k$  matrices in  $ASym_n(\mathbb{C}) := \{X \in M_{2n}(\mathbb{C}) \mid X + X^T = 0\}$ . We denote these three varieties as  $\mathcal{D}_{m,n}^{\leq k}$ ,  $\mathcal{SD}_n^{\leq k}$ , and  $\mathcal{AD}_n^{\leq 2k}$ , respectively.

### 3. PROOF OF THE MAIN THEOREM

**Theorem.** *Let  $\lambda_1, \dots, \lambda_k$  be dominant integral weights. Then*

$$HS_{\mathbf{q}} \langle \lambda_1, \dots, \lambda_k \rangle = \prod_{\alpha \in \Phi^+} \left( 1 + c_{\lambda_1}(\alpha) q_1 \frac{\partial}{\partial q_1} + \dots + c_{\lambda_k}(\alpha) q_k \frac{\partial}{\partial q_k} \right) \prod_{i=1}^k \frac{1}{1-q_i},$$

$$\text{where } c_\lambda(\alpha) := \frac{(\lambda, \alpha)}{(\rho, \alpha)}.$$

*Proof.* By the Weyl Dimension Formula, we have

$$(3.1) \quad HS_{\mathbf{q}} \langle \lambda_1, \dots, \lambda_k \rangle = \sum_{\mathbf{a} \in \mathbb{N}^k} \prod_{\alpha \in \Phi^+} (1 + a_1 c_{\lambda_1}(\alpha) + \dots + a_k c_{\lambda_k}(\alpha)) \mathbf{q}^{\mathbf{a}}.$$

, where  $\mathbf{a} := (a_1, \dots, a_k)$ , and  $\mathbf{q}^{\mathbf{a}} := q_1^{a_1} \dots q_k^{a_k}$ . Consider the product

$$\prod_{\alpha \in \Phi^+} (1 + a_1 c_{\lambda_1}(\alpha) + \dots + a_k c_{\lambda_k}(\alpha)).$$

It is a polynomial in the  $a_i$  for  $1 \leq i \leq k$ . So we have

$$(3.2) \quad \prod_{\alpha \in \Phi^+} (1 + a_1 c_{\lambda_1}(\alpha) + \cdots + a_k c_{\lambda_k}(\alpha)) = \sum_{|\mathbf{i}| \leq d} b_{\mathbf{i}} \mathbf{a}^{\mathbf{i}},$$

where  $d := |\Phi^+|$ , and  $|\mathbf{i}| := i_1 + \cdots + i_k$ . The coefficients do not depend on  $\mathbf{a}$ . Thus, (3.1) becomes

$$(3.3) \quad \sum_{|\mathbf{i}| \leq d} b_{\mathbf{i}} \sum_{\mathbf{a} \in \mathbb{N}^k} \mathbf{a}^{\mathbf{i}} \mathbf{q}^{\mathbf{a}}.$$

We now find a closed form for  $\sum_{\mathbf{a} \in \mathbb{N}^k} \mathbf{a}^{\mathbf{i}} \mathbf{q}^{\mathbf{a}}$ . Note that if we define

$$f_{(i_1, \dots, i_k)}(\mathbf{q}) := \sum_{\mathbf{a} \in \mathbb{N}^k} \mathbf{a}^{\mathbf{i}} \mathbf{q}^{\mathbf{a}},$$

then hitting  $f_{(i_1, \dots, i_k)}(\mathbf{q})$  with the partial differential operator  $q_j \frac{\partial}{\partial q_j}$  gives us

$f_{(i_1, \dots, i_j+1, \dots, i_k)}(\mathbf{q})$ . Since  $f_{(0, \dots, 0)}(\mathbf{q}) = \prod_{j=1}^k \frac{1}{1 - q_j}$ , and these operators commute, we have

$$f_{(i_1, \dots, i_k)}(\mathbf{q}) = \left( q_1 \frac{\partial}{\partial q_1} \right)^{i_1} \cdots \left( q_k \frac{\partial}{\partial q_k} \right)^{i_k} \prod_{j=1}^k \frac{1}{1 - q_j} = \left( \frac{\partial}{\partial \mathbf{q}} \right)^{\mathbf{i}} \prod_{j=1}^k \frac{1}{1 - q_j}.$$

Therefore, (3.3) becomes

$$(3.4) \quad \sum_{|\mathbf{i}| \leq d} b_{\mathbf{i}} \left( \frac{\partial}{\partial \mathbf{q}} \right)^{\mathbf{i}} \prod_{j=1}^k \frac{1}{1 - q_j}.$$

Then, we have

$$\sum_{|\mathbf{i}| \leq d} b_{\mathbf{i}} \left( \frac{\partial}{\partial \mathbf{q}} \right)^{\mathbf{i}} = \prod_{\alpha \in \Phi^+} \left( 1 + c_{\lambda_1}(\alpha) \frac{\partial}{\partial q_1} + \cdots + c_{\lambda_k}(\alpha) \frac{\partial}{\partial q_k} \right),$$

since this is just (3.2) with the substitution  $a_i \mapsto \frac{\partial}{\partial q_i}$ . The result follows.  $\square$

#### 4. EXAMPLES

By setting  $k = 1$ , we obtain the Hilbert series of an equivariant embedding of a projective variety, as in the case of [6]. If  $G$  has rank  $k$ , and we look at the formal power series given by the fundamental dominant weights  $\langle \omega_1, \dots, \omega_k \rangle$ , we obtain a generating function for the dimensions of the irreducible representations of  $G$ , as in [2].

Inside our given Borel subgroup  $B \subset G$ , we have a maximal unipotent subgroup  $U$  such that  $B = TU$ . The quotient  $G/U$  has a natural structure as an affine variety. This variety has coordinate ring

$$(4.1) \quad \mathbb{C}[G/U] \cong \bigoplus_{\lambda \in P_+(\mathfrak{g})} \mathbb{C}_{\lambda} \otimes L(\lambda),$$

(see, for example, §3.3 in [8]). If we set  $V(\lambda) := \mathbb{C}_\lambda \otimes L(\lambda)$ , we have a gradation on  $\mathbb{C}[G/U]$  given by  $V(\lambda)V(\mu) = V(\lambda + \mu)$ . Then replacing  $P_+(\mathfrak{g})$  with a lattice cone  $\langle \lambda_1, \dots, \lambda_k \rangle$  gives a subalgebra of  $\mathbb{C}[G/U]$ . The spectrum of this subalgebra is a variety, and this variety has an  $\mathbb{N}^k$ -graded Hilbert series given by  $HS_{\mathbf{q}}\langle \lambda_1, \dots, \lambda_k \rangle$ .

Our main interest in examples is going to be using the formula from the main theorem to find a series in  $k$  variables and then specializing that series to a Hilbert series on the underlying variety given by subalgebras of (4.1) corresponding to a given lattice cone  $\langle \lambda_1, \dots, \lambda_k \rangle$ .

Some interesting examples are those given by looking at the homogeneous coordinate ring of the three determinantal varieties  $\mathcal{D}_{m,n}^{\leq k}$ ,  $\mathcal{SD}_n^{\leq k}$ , and  $\mathcal{AD}_n^{\leq 2k}$ . We begin with the symmetric determinantal varieties  $\mathcal{SD}_n^{\leq k}$ . Note that finding the Hilbert series for these varieties is in general a very difficult thing to do (see, for example, [3], [4]).

The Second Fundamental Theorem of Invariant Theory for  $O(n)$  (see, for example, [5], p.561), states that the homogeneous coordinate ring  $\mathcal{SD}_n^{\leq k}$  decomposes as an  $SL(n, \mathbb{C})$ -module in the following way:

$$\mathbb{C}[\mathcal{SD}_n^{\leq k}] \cong \bigoplus_{\lambda} L(\lambda),$$

where  $\lambda$  runs over all even dominant integral weights of depth at most  $k$ . Here, an even weight of depth at most  $k$  is one that lies in the lattice cone  $\langle 2\omega_1, \dots, 2\omega_k \rangle$ , where  $\omega_1, \dots, \omega_{n-1}$  are the fundamental dominant weights of  $SL(n, \mathbb{C})$ , and we are using the standard Borel subgroup of upper triangular matrices in  $SL(n, \mathbb{C})$ .

So we can compute the series  $HS_{\mathbf{q}}\langle 2\omega_1, \dots, 2\omega_k \rangle$  and specialize the variables in an appropriate way to recover the Hilbert series of the standard embedding of the symmetric determinantal variety. The standard Hilbert series on  $\mathcal{SD}_n^{\leq k}$  is given by

$$\sum_{\lambda} \dim(L(\lambda)) q^{|\lambda|},$$

where again,  $\lambda$  runs over all even dominant integral weights of depth at most  $k$ . After computing the series  $HS_{\mathbf{q}}\langle 2\omega_1, \dots, 2\omega_k \rangle$ , we specialize to the standard Hilbert series by making the substitution  $q_i \mapsto q^i$  for  $i = 1, \dots, k$ .

We now compute some examples. We consider the variety  $\mathcal{SD}_4^{\leq 2}$ . Then we compute the series  $HS_{\mathbf{q}}\langle 2\omega_1, 2\omega_2 \rangle$ , where  $\omega_1$  and  $\omega_2$  are the first two fundamental dominant weights of  $SL(4, \mathbb{C})$ . The main theorem gives us the following closed form for  $HS_{\mathbf{q}}\langle 2\omega_1, 2\omega_2 \rangle$ :

$$\prod_{1 \leq i < j \leq 4} \left( 1 + 2c_{\omega_1}(\epsilon_i - \epsilon_j)q_1 \frac{\partial}{\partial q_2} + 2c_{\omega_2}(\epsilon_i - \epsilon_j)q_2 \frac{\partial}{\partial q_2} \right) \frac{1}{(1 - q_1)(1 - q_2)},$$

where  $\Phi^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq 4\}$ , and  $\epsilon_i$  is the functional that gives the  $i$ th diagonal element of a matrix in  $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{C})$ . Then computing  $c_{\omega_1}(\epsilon_i - \epsilon_j)$  and  $c_{\omega_2}(\epsilon_i - \epsilon_j)$  for  $1 \leq i < j \leq 4$  gives us

$$(1 + 2q_1 \frac{\partial}{\partial q_1})(1 + 2q_2 \frac{\partial}{\partial q_2})(1 + q_1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial q_2})(1 + q_2 \frac{\partial}{\partial q_2})(1 + \frac{2}{3}q_1 \frac{\partial}{\partial q_1} + \frac{2}{3}q_2 \frac{\partial}{\partial q_2}) \frac{1}{(1 - q_1)(1 - q_2)}.$$

Applying the differential operators then yields

$$\frac{1 + 6q_1 + 15q_2 + q_1^2 + 16q_1q_2 + 15q_2^2 + q_3^3 - 50q_1q_2^2 - 29q_1^2q_2 - 4q_1q_2^3 - 25q_1^2q_2^2 + 6q_1^3q_2 + 21q_1^2q_2^3 + 20q_1^3q_2^2 + 6q_1^3q_2^3}{(1 - q_1)^4(1 - q_2)^5}.$$

This formula seems unmanageable, but is easy to compute with Mathematica or Maple, and after we make the substitution  $q_i \mapsto q^i$ , we get

$$\frac{1 + 3q + 6q^2}{(1 - q)^7},$$

which is the Hilbert series for the standard embedding of  $\mathcal{SD}_4^{\leq 2}$ .

We can then increase the size of the matrices to get a recursive way of finding the Hilbert series of  $\mathcal{SD}_n^{\leq 2}$ . Let  $\{\alpha_1, \dots, \alpha_{n-1}\}$  be the simple roots of  $SL(n, \mathbb{C})$ . The only positive roots of  $SL(n, \mathbb{C})$  that contribute to the product in  $HS_{\mathbf{q}}\langle\omega_1, \omega_2\rangle$ , are those which can be written as a string of simple roots  $\sum \alpha_i$  beginning at either  $\alpha_1$  or  $\alpha_2$ . So, as we go from  $n - 1$  to  $n$ , we add two differential operators, namely, those which correspond to the positive roots  $\alpha_2 + \dots + \alpha_{n-1}$  and  $\alpha_1 + \dots + \alpha_{n-1}$ . If we define  $HS_{\mathbf{q}}^n\langle\omega_1, \omega_2\rangle$  to be the series given by the first two fundamental dominant weights of  $SL(n, \mathbb{C})$ , we have the following recursive formula.

**Lemma.** For  $n \geq 3$ ,

$$HS_{\mathbf{q}}^n\langle 2\omega_1, 2\omega_2 \rangle = (1 + \frac{2}{n-2}q_2 \frac{\partial}{\partial q_2})(1 + \frac{2}{n-1}q_1 \frac{\partial}{\partial q_1} + \frac{2}{n-1}q_2 \frac{\partial}{\partial q_2})HS_{\mathbf{q}}^{n-1}\langle 2\omega_1, 2\omega_2 \rangle.$$

We obtain the recursion by simply computing the coefficients for the two new weights. Note that this is a linear recursion on the multi-variate series, but it does not pass to a recursion on the single variable Hilbert series for the varieties  $\mathcal{SD}_n^{\leq 2}$ . In this way, the multi-variate series behaves more nicely than the single variable Hilbert series. This multi-variate series then allows us to more easily compute the Hilbert series of  $\mathcal{SD}_n^{\leq k}$ .

We have a similar story for the Hilbert series of the standard embedding of  $\mathcal{AD}_n^{\leq 2k}$ . The Second Fundamental Theorem of Invariant Theory for  $Sp(2n, \mathbb{C})$  (see, for example, p. 562 in [5]) says that the homogenous coordinate ring of  $\mathcal{AD}_n^{\leq 2k}$  decomposes as an  $SL(2n, \mathbb{C})$ -module as

$$\mathbb{C}[\mathcal{AD}_n^{\leq 2k}] \cong \sum_{\lambda} L(\lambda),$$

where  $\lambda$  runs over the lattice cone  $\langle\omega_2, \omega_4, \dots, \omega_{2k}\rangle$ . Then  $HS_{\mathbf{q}}\langle\omega_2, \omega_4, \dots, \omega_{2k}\rangle$  can again be specialized to the standard Hilbert series given by

$$\sum_{\lambda} \dim(L(\lambda))q^{|\lambda|},$$

where  $\lambda$  runs over  $\langle\omega_2, \omega_4, \dots, \omega_{2k}\rangle$ , by making the substitution  $q_i \mapsto q^i$ .

We finish with an example beyond the scope of the determinantal varieties. We consider the lattice cone  $\langle 3\omega_1, 3\omega_2 \rangle$  in the weight lattice  $P_+(\mathfrak{sl}(3, \mathbb{C}))$ . This series corresponds to the subalgebra

$$\bigoplus_{\lambda \in \langle 3\omega_1, 3\omega_2 \rangle} \mathbb{C}_{\lambda} \otimes L(\lambda)$$

of (4.1). If we take the torus  $T \cong (\mathbb{C}^{\times})^2$  of  $SL(3, \mathbb{C})$ , then this algebra corresponds to a coordinate of the variety  $G/AU$ , where  $A$  is the finite subgroup of  $T$  generated by  $\{(\zeta_3, 1), (1, \zeta_3)\}$ , where  $\zeta_3$  is a primitive third root of unity.

From (1.2), we have the following closed form for  $HS_{\mathbf{q}}\langle 3\omega_1, 3\omega_2 \rangle$ :

$$\prod_{1 \leq i < j \leq 3} \left( 1 + 3c_{\omega_1}(\epsilon_i - \epsilon_j)q_1 \frac{\partial}{\partial q_1} + 3c_{\omega_2}(\epsilon_i - \epsilon_j)q_2 \frac{\partial}{\partial q_2} \right) \frac{1}{(1 - q_1)(1 - q_2)},$$

where  $\Phi^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq 3\}$ . Then computing the coefficients yields

$$(1 + 3q_1 \frac{\partial}{\partial q_1})(1 + 3q_2 \frac{\partial}{\partial q_2})(1 + \frac{3}{2}q_1 \frac{\partial}{\partial q_1} + \frac{3}{2}q_2 \frac{\partial}{\partial q_2}) \frac{1}{(1 - q_1)(1 - q_2)},$$

and computing the partial derivatives yields

$$\frac{1 + 7q_1 + 7q_2 + q_1^2 + q_2^2 + 13q_1q_2 - 11q_1^2q_2 - 11q_1q_2^2 + 8q_1^2q_2^2}{(1 - q_1)^3(1 - q_2)^3}.$$

We can perform various substitutions to find nice formulas for Hilbert series of different embeddings of  $G/AU$ . For instance, after performing the substitution  $q_i \mapsto q$ , we get

$$\frac{1 + 14q + 15q^2 - 22q^3 + 8q^4}{(1 - q)^6}.$$

We reduce the complexity in performing the calculation of the above Hilbert series by first computing the series in two variables, and then specializing it to a Hilbert series for  $G/AU$ .

The series  $HS_{\mathbf{q}}(\lambda_1, \dots, \lambda_k)$  is useful in computing these series, as it bypasses much of the complicated machinery normally used in their computation. The multi-variate series applies to any variety whose coordinate ring can be decomposed into highest weight representations whose weights run over a lattice cone in  $P_+(\mathfrak{g})$ . These include the three classes of determinantal varieties, but also include many other interesting examples, such as the flag variety  $G/U$ .

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN–MILWAUKEE  
*E-mail address*: waj@uwm.edu